SIGNALS AND SYSTEMS - WEEK 8

Problem 1

Find the region of convergence and Laplace transforms of the functions below.

Sol

Region of convergence: Value of $Re\{s\} = \sigma$ that makes

$$\int_{-\infty}^{\infty} x(t) e^{-st}\, dt \text{ convergent.}$$

For signals that are not everlasting Region of convergence is every value of σ (entire s-plane).

a) $u(t) - u(t-1)$.

Non-everlasting signal (finite duration), so ROC = s-plane.

$$\mathcal{L}\{u(t)-u(t-1)\} = \int_{-\infty}^{\infty} [u(t)-u(t-1)] e^{-st}\, dt = \frac{1}{s}(1-e^{-s})$$

b) $te^{-t} u(t)$

ROC: $$te^{-t} e^{-st} u(t) = te^{-t(1+s)} u(t) = \underbrace{te^{-t(1+\sigma)}}_{\text{Must decrease to converge}} e^{-j\omega t} u(t)$$

$Re\{s\} = \sigma > -1$

$$\mathcal{L}\{te^{-t} u(t)\} = \int_{-\infty}^{\infty} te^{-t} e^{-st} u(t)\, dt = \frac{1}{(s+1)^2}$$

c) $t\cos(\omega_0 t)\,u(t)$

ROC: $t\cos(\omega_0 t)e^{-st}u(t) = \underbrace{t\cos(\omega_0 t)e^{-\sigma t}}_{\text{Must decrease}} e^{-j\omega t}\,u(t)$

$Re\{s\} = \sigma > 0$

$$\mathcal{L}\{t\cos(\omega_0 t)\,u(t)\} = \int_{0^-}^{\infty} t\cos(\omega_0 t)\,u(t)e^{-st}\,dt = \frac{s^2 - \omega_0^2}{(s^2+\omega_0^2)^2}$$

d) $(e^{+2t} - 2e^{-t})u(t)$

ROC: $[e^{2t} - 2e^{-t}]u(t)\cdot e^{-st} = [\underbrace{e^{-t(-2+s)}}_{\text{Important term}} - 2e^{-t(1+s)}]u(t)$

ROC: $Re\{s\} = \sigma > 2$

$$\mathcal{L}\{[e^{2t} - 2e^{-t}]u(t)\} = \int_{0^-}^{\infty} [e^{2t} - 2e^{-t}]e^{-st}\,u(t)\,dt = \frac{1}{s-2} - \frac{2}{s+1}$$

Region of convergence is all values bigger than σ.
If the $j\omega$-axis is not inside ROC the Fourier Transform doesn't exist for that signal.

$\mathcal{F}\{[e^{2t} - 2e^{-t}]u(t)\} \rightarrow$ does not exist.

Problem 2

Find the inverse Laplace transforms of the functions below.

Sol

a) $\frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)}$ $\rightarrow$ $s=-2$, $s=-3$ are poles

Convert to partial fraction:

$$k_1 = \frac{2\cdot(-2)+5}{-2+3} = \frac{-4+5}{-2+3} = \frac{1}{1} = 1$$

$$k_2 = \frac{2\cdot(-3)+5}{-3+2} = \frac{-6+5}{-1} = \frac{-1}{-1} = 1$$

Partial fraction: $\frac{k_1}{s+2} + \frac{k_2}{s+3} = \frac{1}{s+2} + \frac{1}{s+3}$

Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}+\frac{1}{s+3}\right\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}}_{\text{Lookup table 5}} = \left[e^{-2t}+e^{-3t}\right]u(t)$$

b) $\frac{3s+5}{s^2+4s+13}$

$$\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2+4s+13}\right\} = e^{-2t}\left[3\cos(3t) - \frac{1}{3}\sin(3t)\right]u(t)$$

↑ Table entry 10d

c) $\frac{(s+1)^2}{s^2-s-6}$

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{s^2-s-6}\right\} = \delta(t) + \frac{16}{5}e^{3t} - \frac{1}{5}e^{-2t}$$

↑ Highpass filter