SIGNALS AND SYSTEMS - WEEK 5

Problem 1

Calculate the Fourier Transform of the functions.

Sol

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{0} dt = e^{0} \int_{-\infty}^{\infty} \delta(t) dt = e^{0} = 1$$

Two properties:

- $f(t)\delta(t) = f(0)\delta(t)$
- $\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0)$

$$\mathcal{F}\{\delta(\omega)\}^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{e^{0}}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) d\omega = \frac{1}{2\pi}$$

$$\mathcal{F}\{2\pi\delta(\omega)\}^{-1} = 2\pi \cdot \mathcal{F}\{\delta(\omega)\}^{-1} = 1$$

$$\mathcal{F}\{2\pi\delta(\omega-\omega_0)\}^{-1} = \frac{2\pi}{2\pi} \cdot \int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \int \delta(\omega-\omega_0) d\omega = e^{j\omega_0 t}$$

$$\mathcal{F}\{\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]\}^{-1} = \frac{\pi}{2\pi} \left[\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega + \int \delta(\omega+\omega_0) e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2} \cdot \left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right] = \frac{1}{2} \left[\cos(\omega_0 t) + j\sin(\omega_0 t) + \cos(\omega_0 t) - j\sin(\omega_0 t) \right] = \frac{1}{2} \cdot 2\cos\omega_0 t$$

$$= \cos(\omega_0 t)$$

Problem 3

$x_e(t) = e^{-4|t|}$ and $x_o(t) = -e^{-4|t|} \cdot u(-t) + e^{-4|t|} \cdot u(t)$

What symmetries do you expect in the Fourier Transforms.

Calculate the Fourier Transforms.

Sol

Any signal can be split into an even and odd part.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} (x_e + x_o)(\cos(\omega_o t) - j\sin(\omega_o t))$$

$$= \int_{-\infty}^{\infty} x_e(t) \cos(\omega t)\, dt + j\int_{-\infty}^{\infty} -x_e(t) \sin(\omega t)\, dt + \int_{-\infty}^{\infty} x_o(t) \cos(\omega t)\, dt - j\int x_o(t) \sin \omega t$$

$$= \int_{-\infty}^{\infty} x_e(t) \cos(\omega t)\, dt + j\int -x_o(t) \sin(\omega t)\, dt$$

$x_e(t)$ is purely even (no $x_o(t)$)

$x_o(t)$ is purely odd (no $x_e(t)$)

Expectations:

$X_e(\omega) = \mathcal{F}\{x_e(t)\} \longrightarrow$ Real and even

$X_o(\omega) = \mathcal{F}\{x_o(t)\} \longrightarrow$ Imaginary and odd.

Look at $x_e(t) = e^{-4|t|}$ first.

$$x_e(t) = \begin{cases} e^{4t} & t < 0 \\ e^{-4t} & t > 0 \end{cases}$$

$$\mathcal{F}\{x_e(t)\} = \int_{-\infty}^{\infty} x_e(t) e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{4t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-4t} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(4-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-4-j\omega)t}\, dt = \frac{1}{4-j\omega}\left[e^{(4-j\omega)t}\right]_{-\infty}^{0} + \frac{1}{-4-j\omega}\left[e^{(-4-j\omega)t}\right]_{0}^{\infty}$$

$$= \frac{e^0}{4-j\omega} + \frac{-e^0}{-4-j\omega} = \frac{1}{4-j\omega} - \frac{1}{-4-j\omega}$$

Rectangular form:

$$\frac{1}{4-j\omega} = \frac{4+j\omega}{(4-j\omega)(4+j\omega)} = \frac{4+j\omega}{16+\omega^2}$$

$$\frac{1}{-4-j\omega} = \frac{-4+j\omega}{(-4-j\omega)(-4+j\omega)} = \frac{-4+j\omega}{16+\omega^2}$$

Total result: $\frac{4+j\omega}{16+\omega^2} - \frac{-4+j\omega}{16+\omega^2} = \frac{8}{16+\omega^2}$

Real and even $\rightarrow$ matches expectation

Now look at $x_o(t) = -e^{-4|t|}u(t) + e^{-4|t|}u(-t)$.

$$x_o(t) = \begin{cases} -e^{4t} & t<0 \\ e^{-4t} & t>0 \end{cases}$$

$$\mathcal{F}\{x_o(t)\} = \int_{-\infty}^{\infty} x_o(t)e^{-j\omega t}\,dt = \int_{-\infty}^{0} -e^{+4t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-4t}e^{-j\omega t}\,dt$$

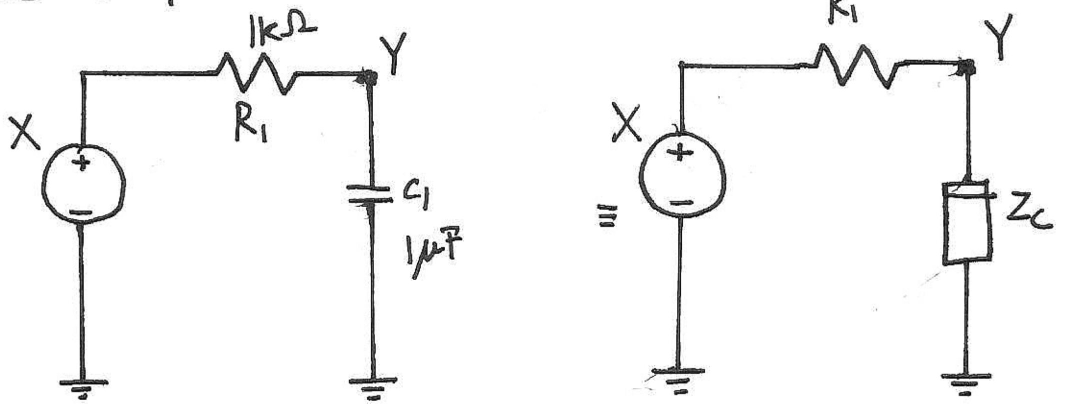
$$= \int_{-\infty}^{0} -e^{(4-j\omega)t}\,dt + \int_{0}^{\infty} e^{(-4-j\omega)t}\,dt = \frac{-1}{4-j\omega}\left[-e^{(4-j\omega)t}\right]_{-\infty}^{0} + \frac{1}{-4-j\omega}\left[e^{(-4-j\omega)t}\right]_{0}^{\infty}$$

$$= -\frac{1}{4-j\omega} + \frac{-1}{-4-j\omega} = \frac{-4-j\omega}{16+\omega^2} + \frac{4-j\omega}{16+\omega^2} = \frac{-2j\omega}{16+\omega^2}$$

Odd and imaginary $\longrightarrow$ matches expectation.

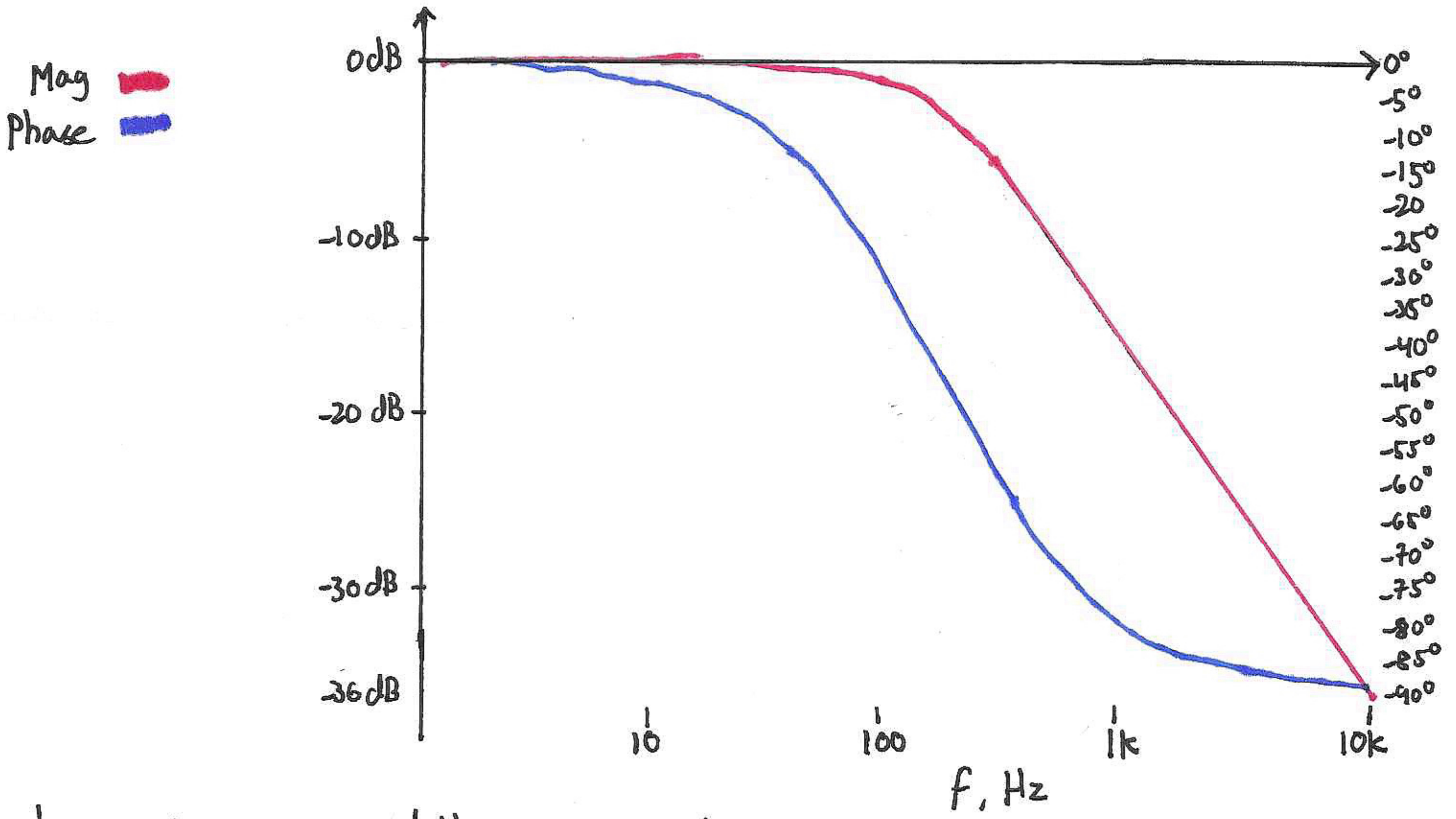
Problem 4

Analyze the simple RC-circuit:



sol

We perform an ac analysis to derive the frequency characteristic $H(j\omega)$.



Clear lowpass filter characteristics.

Transfer function:

$$Y(\omega) = X(\omega) \cdot \frac{Z_C(\omega)}{Z_C(\omega) + R_1} \iff$$

$$H(\omega) = \frac{out}{In} = \frac{Y(\omega)}{X(\omega)} = \frac{Z_C(\omega)}{Z_C(\omega) + R_1}$$

Inserting the impedance for the capacitors.

$$Z_C = \frac{1}{j\omega C}$$

$$H(\omega) = \frac{Z_C}{R_1 + Z_C} = \frac{\frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}} = \frac{1}{j\omega R_1 C + 1}$$

A simple first order lowpass filter.

Magnitude:

$$|H(\omega)| = \frac{|1|}{|j\omega R_1 C + 1|} = \frac{1}{\sqrt{1+\omega^2 R_1^2 C^2}} \rightarrow \text{Even function}$$

Phase:

$$\angle H(\omega) = \angle 1 - \angle (j\omega R_1 C + 1)$$

$$= 0 - \tan^{-1}\left(\frac{\omega R_1 C}{1}\right) = \tan^{-1}\left(\omega R_1 C\right) \rightarrow \text{odd function}$$

Calculating the inverse of transfer function, $H(\omega)$, gives impulse response.

$$\mathcal{F}\{H(\omega)\}^{-1} = \int_{-\infty}^{\infty} \frac{1}{j\omega R_1 C + 1} \cdot e^{j\omega t}\, dt \cdot \frac{1}{2\pi} = 1000 e^{-1000t} u(t)$$

$$h(t) = 1000 e^{-1000t} u(t)$$

where $R_1 = 1k\Omega$, $C = 1\mu F$

Set $R_1 = 500\,\Omega$ and find $H(\omega)$ and $h(t)$.

$R_1 = 1\,k\Omega$: $H(\omega) = \dfrac{1}{j\omega R_1 C + 1}$, $h(t) = 1000 e^{-1000t} u(t)$

$R_1 = 500\,\Omega$: $H(\omega) = \dfrac{1}{j\omega R_1 C + 1}$, $h(t) = 2000 e^{-2000t} u(t)$

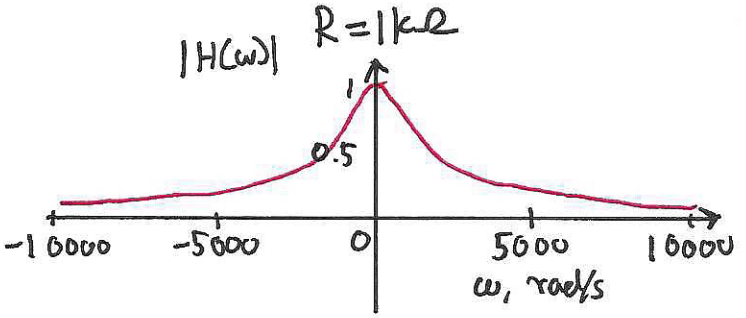
We notice the time scaling property.

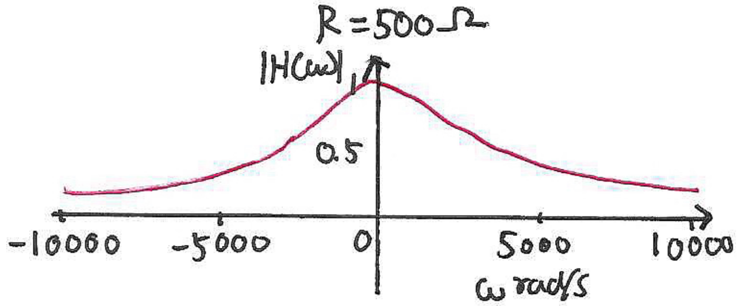
$h(at) \Leftrightarrow \frac{1}{|a|} H\left(\frac{\omega}{a}\right)$

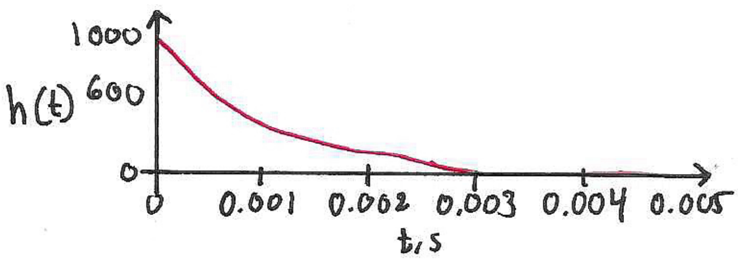
If $h(t)$ decreases faster, $H(\omega)$ becomes wider.

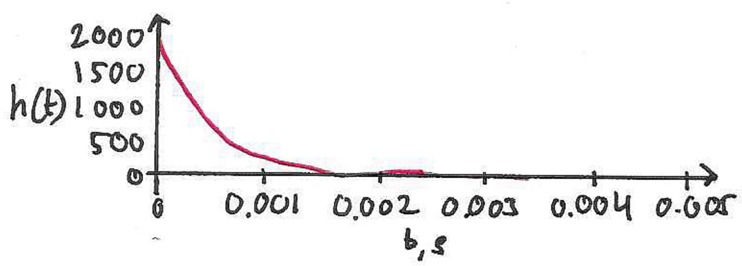
If $h(t)$ decreases slower, $H(\omega)$ becomes more narrow.

- Time scaling bigger than 1 $\rightarrow$ wider spectrum
- Time scaling less than 1 $\rightarrow$ narrower spectrum

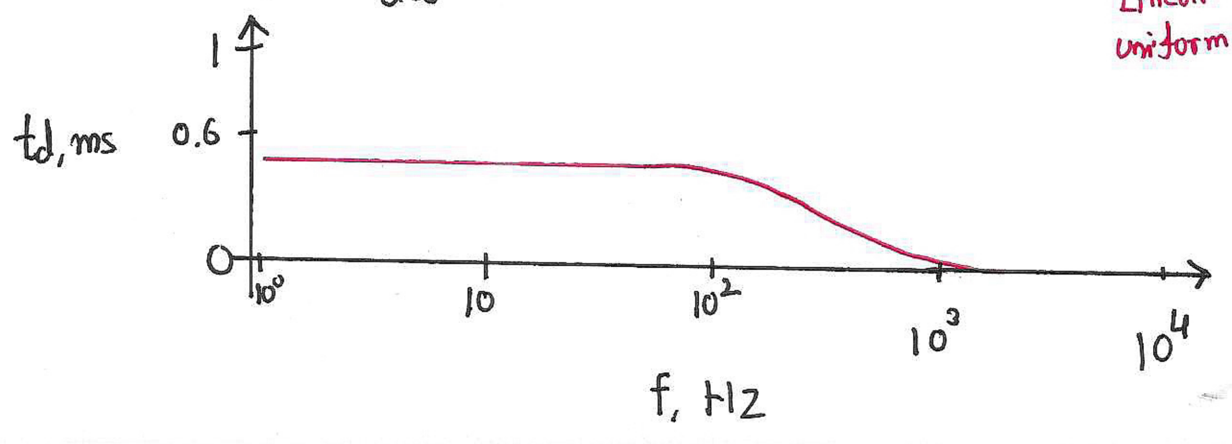








Time delay: $t_d = -\dfrac{d\theta}{d\omega}$



Linear phase = uniform td.