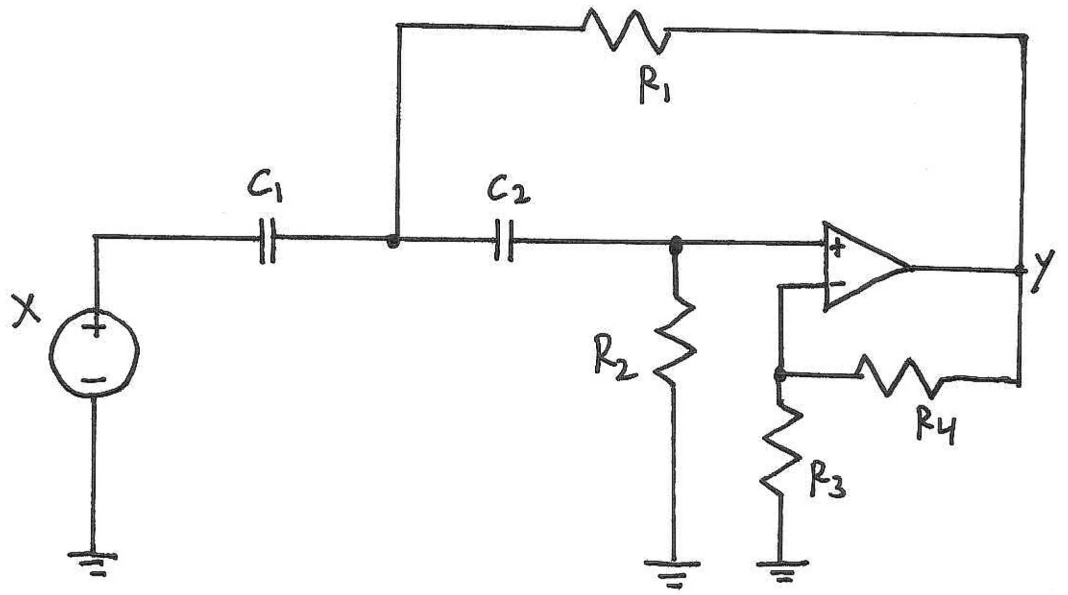
SIGNALS AND SYSTEMS - WEEK 2



Problem 1

Classify the system as linear, time-invariant, causal.

sol

Recall the differential equation:

$$\ddot{y} + 0.6283\dot{y} + 0.0987y = 2\ddot{x}$$

- A system is linear if the terms $y(t)$ and $x(t)$ are linear – that is, there are no $y^2(t)$ or $\sqrt{y(t)}$ or something like that.
- A system is time-invariant if the coefficients of the differential don't depend on time (they are constants).
- A system is causal if the output only depends and past and present values of the input. I.e. there are no $x(t+3)$ or something like that.

Clearly, the highpass filter is LTIC.

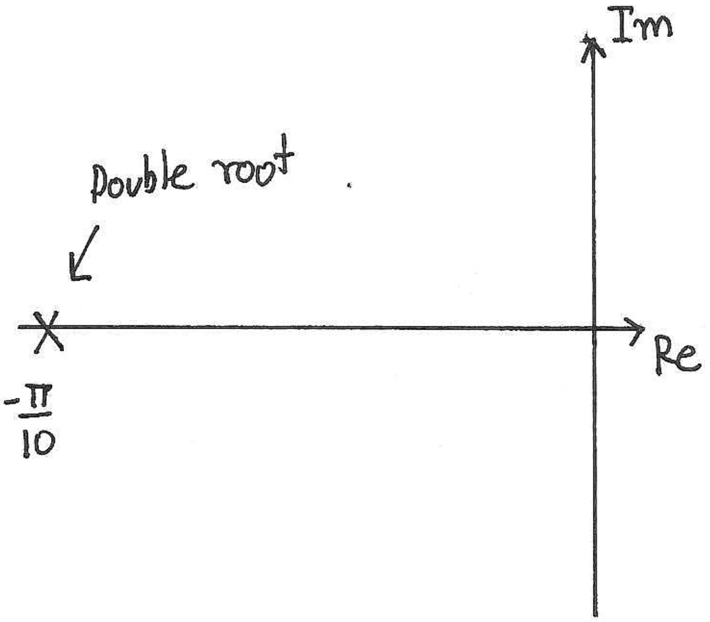
Problem 2

Draw the roots of the characteristic polynomial.

sol

$$\lambda^2 + 0.6283\lambda + 0.0987 = 0 \quad \Leftrightarrow$$

$$\lambda = -\frac{\pi}{10} \quad (am = 2)$$

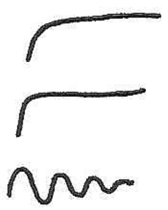


Problem 3

Determine if the system is over, under, or critically damped.

sol

- Distinct real roots $\longrightarrow$ overdamped
- Double root $\longrightarrow$ critically damped ✓
- Complex conjugated roots $\rightarrow$ underdamped



General 2nd order: $\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + y(t)\omega_0^2 = x(t)$

$\omega_0^2 = 0.0987 \Leftrightarrow \omega_0 = 0.314 = \frac{\pi}{10}$

$2\zeta\omega_0 = 0.6283 \Leftrightarrow \zeta = \frac{0.6283}{2\omega_0} = \frac{0.6283}{2 \cdot 0.314} \approx 1 \rightarrow$ Critically damped.

Problem 4

Check if the decomposition property holds.

Sol

Call zero-state response $y_{zs}(t)$ and zero-input response $y_{zi}(t)$.

To check decomposition, the sum of the solutions to

$$Q(D)y_{zi}(t) = 0$$
$$Q(D)y_{zs}(t) = P(D)f(t)$$

should be the same as the solution to

$$Q(D)[y_{zi}(t) + y_{zs}(t)] = P(D)f(t)$$

Let's check our system:

$$\ddot{y} + 0.6283\dot{y} + 0.0987y = 2\ddot{x}$$

$Q(D) = D^2 + 0.6283D + 0.0987$

$P(D) = 2D^2$

$$(D^2 + 0.6283D + 0.0987)y_{zi}(t) = 0$$
$$(D^2 + 0.6283D + 0.0987)y_{zs}(t) = 2D^2 x(t)$$

Adding the two we get

$$(D^2 + 0.6283D + 0.0987)[y_{zi}(t) + y_{zs}(t)] = 2D^2 x(t) \quad (1)$$

From decomposition we set $y(t) = y_{zi}(t) + y_{zs}(t)$ into the original differential equation.

$$Q(D)[y_{zi}(t) + y_{zs}(t)] = P(D)\,x(t)$$
$$(D^2 + 0.6283D + 0.0987)[y_{zi}(t) + y_{zs}(t)] = 2D^2 x(t) \quad (2)$$

Equation (1) = (2), decomposition holds for linear systems.

Example

$$\ddot{y} + 0.6283\dot{y} + 0.0987y = 2\ddot{x} + 1$$

Decomposition does not hold for non-linear systems.

Zero-input response: $(D^2 + 0.6283D + 0.0987)y_{zi}(t) = 1$

Zero-state response: $(D^2 + 0.6283D + 0.0987)y_{zs}(t) = 2D^2x(t) + 1$

Adding the two

$$(D^2 + 0.6283D + 0.0987)[y_{zi}(t) + y_{zs}(t)] = 2D^2x(t) + 2 \qquad (3)$$

Decomposition says this should be equal to

$$(D^2 + 0.6283D + 0.0987)[y_{zi}(t) + y_{zs}(t)] = 2D^2x(t) + 1 \qquad (4)$$

Equation $(3) \neq (4)$, so decomposition does not hold.

Problem 5

Using the initial conditions $y(0^-)=0$ and $\dot{y}(0^-)=0$ show that the homogeneous solution (i.e. the $y_{zi}(t)$) is $y_{hom}(t)=0$.

Sol

From week 1 we found the general solution

$$y_{hom}(t)=(A_1+A_2t)e^{-\frac{\pi}{10}t}$$

• $y_{hom}(0^-)=A_1$

• $\dot{y}_{hom}(0^-)=-\frac{\pi}{10}A_1+A_2$

$$\left.\begin{aligned} A_1&=0 \\ -\tfrac{\pi}{10}A_1+A_2&=0 \end{aligned}\right\} \begin{aligned} A_1&=0 \\ A_2&=0 \end{aligned}$$

$$y_{hom}(t)=0$$

No zero-input response if initial conditions are zero.

Problem 6

Define the initial conditions for obtaining the impulse response.

Sol

$n=2$: $y_n(0)=0$ and $\dot{y}_n(0)=1$

Lathi pp. 116.

Problem 7

Find the impulse response and plot it.

sol

$$\ddot{y} + 0.6283\dot{y} + 0.0987y = \underbrace{2\ddot{x}}_{P(D) = 2D^2}$$

$a_2 = 1$

$a_1 = 0.6283$

$a_0 = 0.0987$

$b_2 = 2$

$b_1 = 0$

$b_0 = 0$

The differential equation is order $n=2$.

General formula:

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t)$$

First we find $y_n(t)$ using $y(0) = 0$, $\dot{y}(0) = 1$.

$$y_n(t) = (A_1 + A_2 t)e^{-\frac{\pi}{10}t}$$

$$\left.\begin{aligned} \dot{y}_n(0) &= -\frac{\pi}{10}A_1 + A_2 = 1 \\ y_n(0) &= A_1 = 0 \end{aligned}\right\} \begin{aligned} A_1 &= 0 \\ A_2 &= 1 \end{aligned} \Rightarrow y_n(t) = te^{-\frac{\pi}{10}t}, \quad t>0$$

$$h(t) = b_2\delta(t) + [2D^2 y_n(t)]u(t)$$

$$2D^2 y_n(t) = \ddot{y}_n(t) \cdot 2 = -\frac{4\pi}{10}e^{-\frac{\pi}{10}t} + \frac{2\pi^2}{100}te^{-\frac{\pi}{10}t}$$

so,

$$h(t) = 2\delta(t) + \left(-\frac{4\pi}{10} + \frac{2\pi^2}{100}t\right)e^{-\frac{\pi}{10}t} \cdot u(t)$$

Impulse response plot: