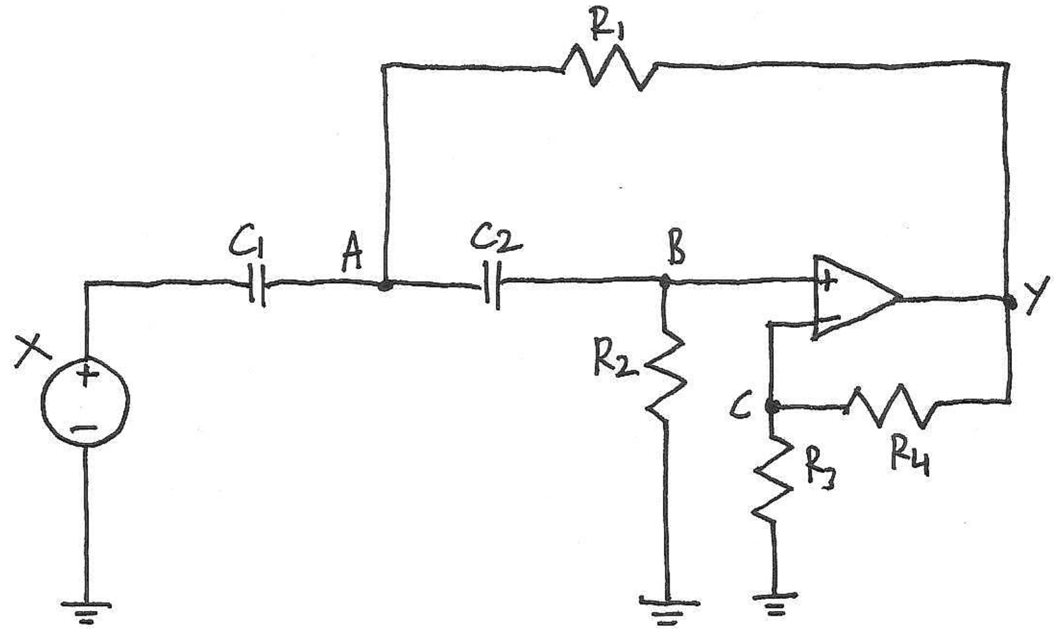
SIGNALS AND SYSTEMS — WEEK 1

Problem 1

Write a set of node equations for this filter:



sol

$$C_1(\dot{V}_A - \dot{x}) + C_2(\dot{V}_A - \dot{V}_B) + \frac{V_A - Y}{R_1} = 0$$

$$C_2(\dot{V}_B - \dot{V}_A) + \frac{V_B}{R_2} = 0$$

where we have used that $I_c = C \cdot \frac{dV}{dt}$.

Problem 2

Obtain the differential equation in the form:

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{x} + b_1\dot{x} + b_0 x$$

Sol

Rewrite the node equations to operator form $s = \frac{d}{dt}$ to reduce the number of unknowns

$$sC_1(V_A - x) + sC_2(V_A - V_B) + \frac{V_A - y}{R_1} = 0$$

$$sC_2(V_B - V_A) + \frac{V_B}{R_2} = 0$$

$$V_C = y \cdot \frac{R_3}{R_3 + R_4}$$

The opamp has negative feedback so we can use the principle of a virtual short.

$$V_B = V_C = \frac{y}{k}, \quad k = 1 + \frac{R_4}{R_3}$$

Define system of equations in Maple:

$$sys := \left\{ sC_1(V_A - x) + sC_2(V_A - V_B) + \frac{V_A - y}{R_1} = 0,\ sC_2(V_B - V_A) + \frac{V_B}{R_2} = 0 \right\}$$

$$V_B := \frac{y}{k}$$

solve(sys, {VA, y}) in Maple yields:

$$y = \frac{C_1 C_2 R_1 R_2 k s^2 x}{C_1 C_2 R_1 R_2 s^2 - C_2 R_2 k s + C_1 R_1 s + C_2 R_1 s + C_2 R_2 s + 1}$$

$$y(C_1 C_2 R_1 R_2 s^2 - C_2 R_2 k s + C_1 R_1 s + C_2 R_1 s + C_2 R_2 s + 1) = C_1 C_2 R_1 R_2 k s^2 x$$

Return to $\dot{y}$ form:

$$C_1 C_2 R_1 R_2 \ddot{y} + \dot{y}(-C_2 R_2 k + C_1 R_1 + C_2 R_1 + C_2 R_2) + y = C_1 C_2 R_1 R_2 k \ddot{x}$$

$$\ddot{y} + \dot{y}\left(\frac{1}{C_2 R_2} + \frac{1}{C_1 R_2} + \frac{1-k}{C_1 R_1}\right) + y \frac{1}{C_1 C_2 R_1 R_2} = k\ddot{x}$$

$a_2 = 1$

$$a_1 = \frac{1}{C_2 R_2} + \frac{1}{C_1 R_2} + \frac{1-k}{C_1 R_1} = 6.283 \cdot 10^{-1}$$

$$a_0 = \frac{1}{C_1 C_2 R_1 R_2} = 9.87 \cdot 10^{-2}$$

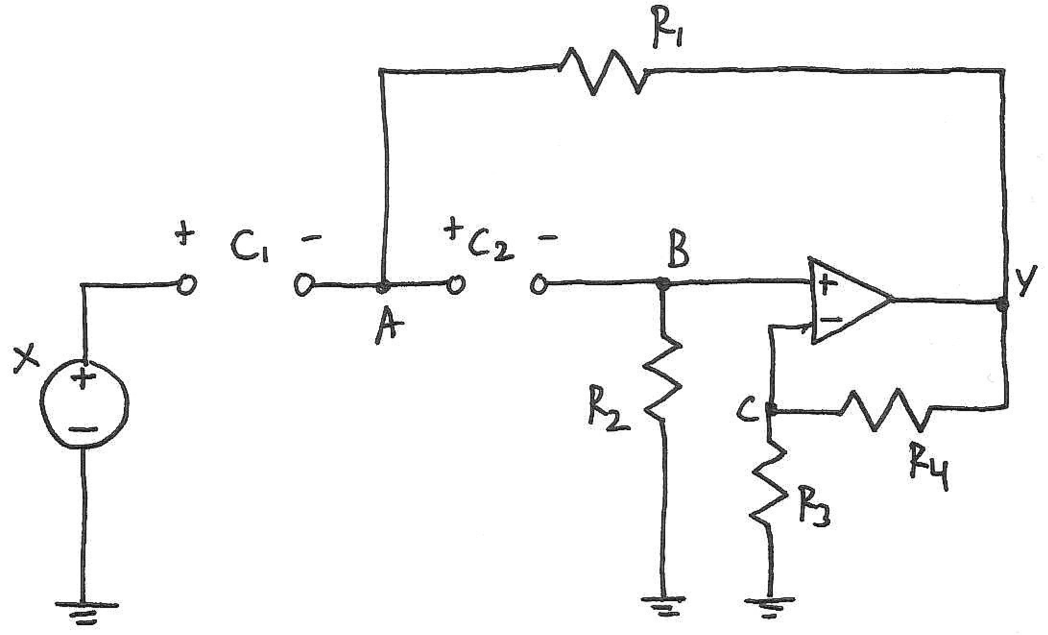
$b_2 = k = 2$

Problem 4

Assume that the input $x = 1V$ for a very long time.
Show that at $t=0^-$ that $V_{C1} = 1\,V$ and $V_{C2} = 0\,V$.

Sol

At DC, capacitors act like open circuits.



No current flows into or out of the opamp terminals.
So, no current flows through $R_2 \Rightarrow V_B = 0$.

$$Y = V_B K = 0$$

No current can flow through R_1, because V_A is surrounded by open circuits.

$$V_A = Y = 0 \qquad \left(\frac{V_A - Y}{R_1} = 0\right)$$

Now the capacitor voltage drops can be evaluated:

$$V_{C1} = X - V_A = 1 - 0 = 1\,V$$

$$V_{C2} = V_B - 0 = 0 - 0 = 0\,V$$

at $t = 0^-$.

Problem 5

Show also that $y(0^-) = 0\,V$ and $\dot{y}(0^-) = 0\,\frac{V}{s}$ by inspection.

Sol

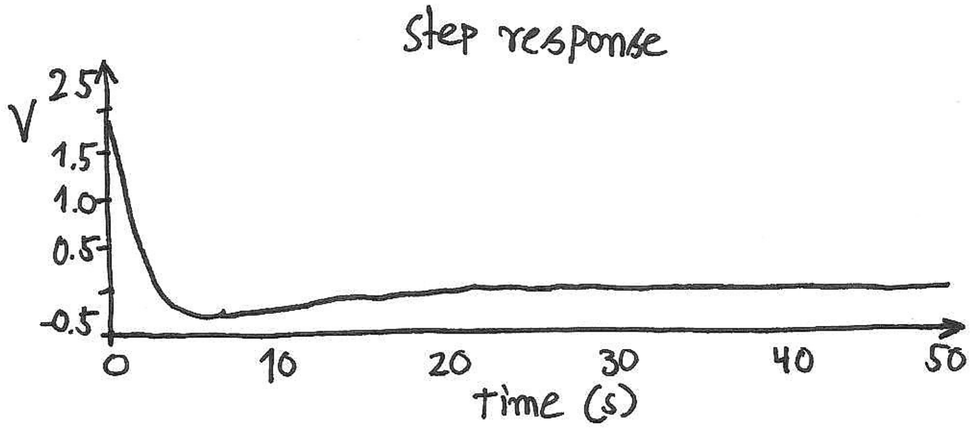
We found that $V_B = 0\,V \Rightarrow y = 0 \cdot k = 0\,V$.

$\dot{y}(0^-)$ must also be $0\,\frac{V}{s}$ because the system has been at rest for a very long time.

Problem 6

Sketch the step response with the initial capacitor voltages set to zero.

Sol

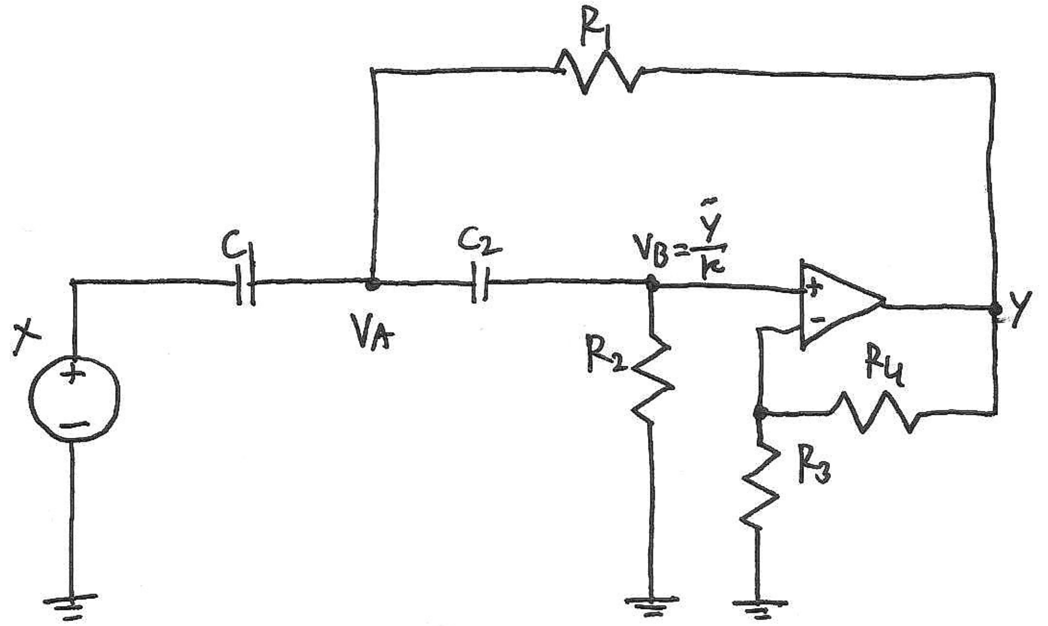


The highpass filter attenuates low frequencies, so the DC-part of the step is not visible on the output.

Problem 7

Assume that $x = 0V$ for a very long time ($V_{C1} = V_{C2} = 0$).
Find the step response analytically, by solving the differential equation.

Sol



The first thing to do is determine $y(0^+)$ and $\dot{y}(0^+)$.
The capacitor voltages cannot change instantaneously, so

$$V_{C1}(0^+) = V_{C1}(0^-) = 0\ V$$

$$V_{C1}(0^+) = x(0^+) - V_A(0^+) = 1V - V_A(0^+) = 0V \Leftrightarrow V_A = 1V.$$

$$V_{C2}(0^+) = V_{C2}(0^-) = 0\ V$$

$$V_{C2}(0^+) = V_A(0^+) - V_B(0^+) = 1V - V_B(0^+) = 0\ V \Leftrightarrow V_B(0^+) = 1V$$

$$y(0^+) = kV_B(0^+) = 2V$$

To determine $\dot{y}(0^+)$ we need an equation, set $\dot{V}_A = V_{Am}$, $\dot{V}_B = \frac{y_m}{k}$ to utilize Maple's solve feature.

$$sys := \left\{ C_1(V_{Am} - x_m) + C_2(V_{Am} - V_{Bm}) + \frac{V_A - y}{R_1} = 0,\ \frac{V_B}{R_2} + (V_{Bm} - V_{Am})C_2 = 0 \right\}$$

Solving the equations in Maple yields:

$$y_m(0^+) = \dot{y}(0^+) = k \cdot \frac{C_1C_2R_1R_2\,\dot{x}(0^+) - V_B(0^+)[C_1R_1 + C_2R_1] - V_A(0^+)C_2R_2 + y(0^+)C_2R_2}{C_1C_2R_1R_2}$$

Inserting values yields:

- $y(0^+) = 2\,V$
- $\dot{y}(0^+) = -1.256\ \frac{V}{s}$

Recall the differential equation

$$\ddot{y} + 0.6283\dot{y} + 0.0987y = 2\ddot{x}$$

Particular solution to step input, $x(t) = u(t)$:

$\ddot{x}(t) = 0$

A particular solution to $\ddot{y} + 0.6283\dot{y} + 0.0987y = 0$ is simply

$$y_p = 0$$

Homogeneous solution:

$$\lambda^2 + 0.6283\lambda + 0.0987 = 0 \Leftrightarrow \lambda = -\frac{\pi}{10} \quad (am = 2)$$

This double root yields general solution:

$$y_{hom}(t) = (A_1 + A_2t)e^{-\frac{\pi}{10}t}$$

$$y_{total}(t) = y_{hom}(t) + y_p(t) = (A_1 + A_2t)e^{-\frac{\pi}{10}t}$$

$y_{total}(0^+) = A_1$

$\dot{y}_{total}(0^+) = A_2 - \frac{\pi}{10}A_1$

$$\left.\begin{aligned} A_1 &= 2 \\ -\frac{\pi}{10}A_1 + A_2 &= -1.256 \end{aligned}\right\} \begin{aligned} A_1 &= 2 \\ A_2 &= -0.628 \end{aligned}$$

Finally

$$y_{total}(t) = (2 - 0.628t)e^{-\frac{\pi}{10}t}, \quad t > 0$$